1. *Calculul unor limite de şiruri folosind integrala Riemann*

Fie f: $[a, b] \to \mathbf{R}$ o funcţie integrabilă pe $[a, b]$. Din definiţia integralei Riemann rezultă că:

$(\forall)\Delta_n = (x_0,...,x_n)$ *cu* $\Delta \in D[a,b]$; $\lim_{n\to\infty}\|\Delta_n\| = 0$ şi orice sistem de puncte intermediare $\xi_i \in [x_{i-1}, x_i]; i = \overline{1,n}$, şirul sumelor Riemann

$\sigma_{\Delta_n}(f,\xi_i) = \sum_{k=1}^{n} f(\xi_i)(x_i - x_{i-1})$ este convergent şi tinde la numărul

$$I = \int_a^b f(x)dx$$

Deci:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \int_a^b f(x)dx. \qquad (1.)$$

Alegem $x_i = a + i\dfrac{b-a}{n}$; $i \in \{1, ..., n\}$ şi $\xi_i = x_i$, atunci relaţia (1.) devine

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} = \int_a^b f(x)dx \qquad (2.)$$

Dacă a=0 şi b=1 relaţia (2.) devine

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n} f\left(\frac{i}{n}\right) = \int_0^1 f(x)dx. \qquad (3.)$$

Exemple:

<u>Exemplul 1.1.</u> Să se calculeze limita şirului cu termenul general

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}.$$

Soluţie:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\frac{1}{1+\frac{i}{n}} = \int_0^1\frac{dx}{1+x} = \ln(1+x) = \ln 2.$$

<u>Exemplul 1.2.</u> Să se calculeze limita şirului cu termenul general

$$a_n = \frac{1}{\sqrt{3n^2+1^2}} + \frac{1}{\sqrt{3n^2+2^2}} + ... + \frac{1}{\sqrt{3n^2+n^2}}$$

Soluţie:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{3n^2+k^2}} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{\sqrt{3+\left(\frac{k}{n}\right)^2}}0 = \int_0^1\frac{dx}{\sqrt{x^2+3}} =$$

$$= \ln\left(x+\sqrt{x^2+3}\right)\Big|_0^1 == \ln 3 - \ln\sqrt{3} = \ln\sqrt{3}.$$

<u>Exemplul 1.3.</u> Să se calculeze limita şirului cu termenul general

$$a_n = \frac{1}{n^2}\left(1\sqrt[n]{e^1} + 2\sqrt[n]{e^2} + \ldots + n\sqrt[n]{e^n}\right)$$

Soluţie: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\frac{1}{n^2}\sum\limits_{k=1}^{n} k\sqrt[n]{e^k}\right) = \lim\limits_{n\to\infty}\frac{1}{n}\left(\sum\limits_{k=1}^{n}\frac{k}{n}e^{\frac{k}{n}}\right) = \int\limits_0^1 xe^x dx = e^x(x-1)\Big|_0^1 = 1$

Exemplul 1.4. Să se calculeze limita şirului cu termenul general

$$a_n = \left(1+\frac{\sqrt{1}}{n\sqrt{n}}\right)\left(1+\frac{\sqrt{2}}{n\sqrt{n}}\right)\ldots\left(1+\frac{\sqrt{n}}{n\sqrt{n}}\right)$$

Soluţie: Fie L= $\lim\limits_{n\to\infty}\left(1+\frac{\sqrt{1}}{n\sqrt{n}}\right)\left(1+\frac{\sqrt{2}}{n\sqrt{n}}\right)\ldots\left(1+\frac{\sqrt{n}}{n\sqrt{n}}\right)$

Prin logaritmare rezultă

$$\ln L = \lim_{n\to\infty}\left[\ln\left(1+\frac{1}{n\sqrt{n}}\right) + \ln\left(1+\frac{\sqrt{2}}{n\sqrt{n}}\right) + \ldots + \ln\left(1+\frac{\sqrt{n}}{n\sqrt{n}}\right)\right];$$

Dearece $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} = 1 \Rightarrow$

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{\sqrt{1}}{n\sqrt{n}}\right)}{\frac{\sqrt{1}}{n\sqrt{n}}} = \lim_{n\to\infty}\frac{\ln\left(1+\frac{\sqrt{2}}{n\sqrt{n}}\right)}{\frac{\sqrt{2}}{n\sqrt{n}}} = \ldots = \lim_{n\to\infty}\frac{\ln\left(1+\frac{\sqrt{n}}{n\sqrt{n}}\right)}{\frac{\sqrt{n}}{n\sqrt{n}}} = 1 \Rightarrow$$

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{\sqrt{1}}{n\sqrt{n}}\right) + \ln\left(1+\frac{\sqrt{2}}{n\sqrt{n}}\right) + \ldots + \ln\left(1+\frac{\sqrt{n}}{n\sqrt{n}}\right)}{\frac{\sqrt{1}}{n\sqrt{n}} + \frac{\sqrt{2}}{n\sqrt{n}} + \ldots + \frac{\sqrt{n}}{n\sqrt{n}}} = 1$$

Cum

$$\lim_{n\to\infty}\left(\frac{\sqrt{1}}{n\sqrt{n}} + \frac{\sqrt{2}}{n\sqrt{n}} + \ldots + \frac{\sqrt{n}}{n\sqrt{n}}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}} = \int_0^1\sqrt{x}dx = \frac{2}{3} \Rightarrow$$

$\ln L = \frac{2}{3} \Rightarrow$ L=$e^{\frac{2}{3}} \Rightarrow \lim\limits_{n\to\infty} a_n = e^{\frac{2}{3}}$.

Exemplul 1.5. Să se calculeze limita şirului cu termenul general

$$a_n = \frac{\pi}{3n}\left[\cos\left(\frac{\pi}{6}+\frac{\pi}{3n}\right) + \cos\left(\frac{\pi}{6}+\frac{2\pi}{3n}\right) + \ldots + \cos\left(\frac{\pi}{6}+\frac{n\pi}{3n}\right)\right]$$

Soluţie:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\pi}{3n}\left[\cos\left(\frac{\pi}{6}+\frac{\pi}{3n}\right) + \cos\left(\frac{\pi}{6}+\frac{2\pi}{3n}\right) + \ldots + \cos\left(\frac{\pi}{6}+\frac{n\pi}{3n}\right)\right] =$$

$$= \lim_{n\to\infty}\frac{\pi}{3n}\sum_{k=1}^{n}\cos\left(\frac{\pi}{6}+\frac{k\pi}{3n}\right) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos x dx = \sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 1 - \frac{1}{2} = \frac{1}{2}$$

Exemplul 1.6. Să se calculeze limita şirului cu termenul general

$$a_n = \frac{1}{n}\left(tg\frac{\pi}{3n} + tg\frac{2\pi}{3n} + ... + tg\frac{n\pi}{3n}\right)$$

Soluţie:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}\left(tg\frac{\pi}{3n} + tg\frac{2\pi}{3n} + ... + tg\frac{n\pi}{3n}\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} tg\frac{k\pi}{n} =$$

$$= \frac{3}{\pi}\lim_{n\to\infty}\frac{\pi}{3n}\sum_{k=1}^{n} tg\frac{k\pi}{3n} = \frac{3}{\pi}\int_0^{\frac{\pi}{3}} \mathrm{tg}x dx = -\frac{3}{\pi}\ln\frac{1}{2} = \frac{3}{\pi}\ln 2$$

Exemplul 1.7. Să se calculeze limita şirului cu termenul general

$$a_n = \frac{1 + \sqrt{2} + \sqrt{3} + ... + \sqrt{n}}{n}$$

Soluţie: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n}\left(1 + \sqrt{2} + ... + \sqrt{n}\right) = \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\sqrt{\frac{k}{n}} = \int_0^1 \sqrt{x}dx = \frac{2}{3}$.

Exemplul 1.8. Să se calculeze limita şirului cu termenul general

$$a_n = \frac{1}{n^2}\left[\sqrt{n^2 + 1^2} + \sqrt{n^2 + 2^2} + ... + \sqrt{2n^2}\right]$$

Soluţie:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\sum_{i=1}^{\infty}\frac{1}{n^2}\sqrt{n^2 + i^2} = \lim_{n\to\infty}\frac{1}{n}\sqrt{1 + \left(\frac{i}{n}\right)^2} = \int_0^1 \sqrt{1 + x^2}dx =$$

$$= \frac{1}{2}\left[x\sqrt{x^2 + 1} + \ln\left(x + \sqrt{x^2 + 1}\right)\right]\Big|_0^1 = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\sqrt{2} + 1\right).$$

Exemplul 1.9. Să se calculeze limita şirului cu termenul general

$$a_n = \frac{\pi}{n}\left(\sin\frac{1\pi}{n} + \sin\frac{2\pi}{n} + ... + \sin\frac{n\pi}{n}\right)$$

Soluţie: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{\pi}{n}\sin\frac{k\pi}{n} = \int_0^{\pi}\sin x dx = -\cos x\Big|_0^{\pi} = 2$

2. *Comparaţie între calculul limitelor unor şiruri de numere cu ajutorul integralei Riemann şi cu ajutorul teoremelor de aflare a limitelor de şiruri*

În cele ce urmează vom arăta avantajul şi dezavantajul aplicării metodei de calculare a limitelor unor şiruri de numere cu ajutorul integralei Riemann faţă de metodele clasice. Pentru aceasta vom raţiona comparativ pe câteva exemple.

Exemplul 2.1. Să se calculeze limita şirului cu termenul general

$$a_n = \frac{\sqrt[n]{n!}}{n}$$

Soluţie: metoda I (folosind teoremele de calculare a limitelor de şiruri). Amintim aici un rezultat cunoscut din teoria limitelor de şiruri. Fie şirul de numere $(a_n)_{n\in\mathbf{N}}$ cu $a_n>0$, $(\forall)n\in\mathbf{N}$. Dacă $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\alpha>0$ atunci $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\alpha$.

Revenind la şirul din exemplul 2.1. notăm $b_n=\frac{n!}{n^n}$,

avem: $$\frac{b_{n+1}}{b_n}=\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!}=\frac{n^n}{(n+1)^n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

Trecând la limită rezultă:

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}$$

$$\lim_{n\to\infty}\sqrt[n]{b_n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}=\frac{1}{e} \quad \text{adică} \quad \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}.$$

metoda II (folosind integrala Riemann). Putem scrie:

$$a_n=\frac{\sqrt[n]{n!}}{n}=\sqrt[n]{\frac{n!}{n^n}}=\sqrt[n]{\frac{1}{n}\frac{2}{n}\dots\frac{n-1}{n}\frac{n}{n}}$$

Logaritmând obţinem: $\ln a_n=\frac{1}{n}\sum\limits_{k=1}^{n}\ln\frac{k}{n}\Rightarrow a_n=e^{\sum_{k=1}^{n}\frac{1}{n}\ln\frac{k}{n}}$, unde trecând la limită, găsim:

$$\lim_{n\to\infty}a_n=e^{\lim\limits_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n}\ln\left(\frac{k}{n}\right)\right]}=e^{\int_0^1\ln x\,dx}=e^{-1}=\frac{1}{e}.$$

Exemplul 2.2. Să se găsească limita şirului cu termenul general

$$a_n=\frac{\sqrt[n]{(n+1)(n+2)\dots(2n)}}{n}$$

Soluţie: metoda I (folosind teoremele asupra şirurilor)

$$a_n=\frac{\sqrt[n]{(n+1)(n+2)\dots(2n)}}{n}=\sqrt[n]{\frac{(n+1)(n+2)\dots(2n)}{n^n}}$$

Notăm $b_n=\frac{(n+1)(n+2)\dots(2n)}{n^n}\Rightarrow b_{n+1}=\frac{(n+2)(n+3)\dots(2n)(2n+1)(2n+2)}{(n+1)^{n+1}}\Rightarrow$

$$\frac{b_{n+1}}{b_n}=\frac{(n+2)(n+3)\dots(2n)(2n+1)(2n+2)}{(n+1)^{n+1}}\frac{n^n}{(n+1)(n+2)\dots(2n)},$$ de

unde $\frac{b_{n+1}}{b_n}=\frac{4+\frac{2}{n}}{1+\frac{1}{n}}\frac{1}{\left(1+\frac{1}{n}\right)^n}\Rightarrow \lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n}=\frac{4}{e}\Rightarrow \lim\limits_{n\to\infty}\sqrt[n]{b_n}=\frac{4}{e}\Rightarrow$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{\sqrt[n]{(n+1)(n+2)...(2n)}}{n}=\frac{4}{e}.$$

metoda II (folosind integrala Riemann)

$a_n=\frac{\sqrt[n]{(n+1)(n+2)...(2n)}}{n}=\sqrt[n]{\frac{n+1}{n}\frac{n+2}{n}...\frac{2n}{n}}$ de unde prin logaritmare rezultă

$\ln a_n=\frac{1}{n}\sum\limits_{k=1}^{n}\ln\frac{n+k}{n}=\frac{1}{n}\sum\limits_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)$;

trecând la limită se obţine

$$\lim_{n\to\infty}a_n=e^{\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n}\ln\left(1+\frac{k}{n}\right)}=e^{\int_0^1\ln(1+x)dx}=e^{\ln\frac{4}{e}}=\frac{4}{e}.$$

<u>Exemplul 2.3.</u> Să se calculeze limita şirului cu termenul general

$$a_n=\frac{\pi}{n}\left(\cos^2\frac{1\pi}{2n}+\cos^2\frac{2\pi}{2n}+...+\cos^2\frac{n\pi}{2n}\right)$$

Soluţie: metoda I (folosind teoremele asupra şirurilor). Vom stabili mai întâi formula:

$$\sum_{k=1}^{n}\cos(k\alpha)=\frac{\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

Din relaţiile:

$$\sin\frac{\alpha}{2}\cos\alpha=\frac{1}{2}\left[\sin\frac{3\alpha}{2}-\sin\frac{\alpha}{2}\right]$$

$$\sin\frac{\alpha}{2}\cos 2\alpha=\frac{1}{2}\left[\sin\frac{5\alpha}{2}-\sin\frac{3\alpha}{2}\right]$$

. . .

$$\sin\frac{\alpha}{2}\cos n\alpha=\frac{1}{2}\left[\sin\frac{(2n+1)\alpha}{2}-\sin\frac{(2n-1)\alpha}{2}\right]$$

Prin adunare rezultă:

$$\sin\frac{\alpha}{2}\sum_{k=1}^{n}\cos(k\alpha)=\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2},\Rightarrow$$

$$\sum_{k=1}^{n}\cos(k\alpha)=\frac{\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}.$$ Atunci:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\frac{\pi}{n}\sum_{k=1}^{n}\cos^2\left(\frac{k\pi}{2n}\right)\right] = \lim_{n\to\infty}\left\{\frac{\pi}{2n}\sum_{k=1}^{n}\left[1+\cos\left(\frac{k\pi}{n}\right)\right]\right\} =$$

$$= \lim_{n\to\infty}\left[\frac{\pi}{2}+\frac{\pi}{2n}\sum_{k=1}^{n}\cos\left(\frac{k\pi}{n}\right)\right] = \frac{\pi}{2} + \lim_{n\to\infty}\frac{\sin\frac{\pi}{2}\cos\left(1+\frac{1}{n}\right)\frac{\pi}{2}}{\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}} = \frac{\pi}{2}\ .$$

metoda II (folosind integrala Riemann)

$$a_n = \frac{\pi}{n}\sum_{k=1}^{n}\cos^2\left(\frac{k\pi}{n}\right)$$

$$\lim_{n\to\infty} a_n = 2\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\frac{\pi}{2}}{n}\cos^2\left(\frac{k}{n}\frac{\pi}{2}\right) = 2\int_0^{\frac{\pi}{2}}\cos^2 x\,dx = \frac{\pi}{2}$$

<u>Exemplul 2.4.</u> Să se găsească limita şirului cu termenul general

$$a_n = \frac{1}{n}\left[\left(a+\frac{n+p}{n}\right)^2 + \left(a+\frac{n+p+1}{n}\right)^2 + \ldots + \left(a+\frac{2n+p}{n}\right)^2\right]$$

Soluţie: metoda I

Vom utiliza formulele cunoscute:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2};\quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$a_n = \frac{1}{n}\sum_{k=p}^{n-p}\left[(a+1)+\frac{k}{n}n\right]^2 = \frac{1}{n}\sum_{k=p}^{n-p}\left[(a+1)^2+\frac{2(a+1)}{n}k+\frac{k^2}{n^2}\right] =$$

$$= \frac{1}{n}(a+1)^2(n-2p+1) + \frac{2(a+1)}{n^2}\sum_{k=p}^{n-p}k + \frac{1}{n^3}\sum_{k=p}^{n-p}k^2 =$$

$$= \frac{1}{n}(a+1)^2(n-2p+1) + \frac{2(a+1)}{n^2}\frac{(p+n-p)(n-2p+1)}{2} +$$

$$+\frac{2n^3+(1-6p)n^2+(6p^2-2p-1)n}{6n^3} + \frac{4p^2(1-p)}{6n^3}$$

Deci:

$$a_n = (a+1)^2\left(1+\frac{2p-1}{n}\right) + (a+1)\left(1-\frac{2p-1}{n}\right) +$$

$$+\frac{1}{6}\left[2+\frac{1-6p}{n}+\frac{6p^2-2p-1}{n^2}+\frac{4p^2(1-p)}{n^3}\right]$$

Trecând la limită găsim:

$$\lim_{n\to\infty} a_n = a^2+3a+\frac{7}{3}.$$

metoda II

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{p-1}\left[(a+1)+\frac{k}{n}\right]^2\right\} + \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=p}^{n-p}\left[(a+1)+\frac{k}{n}\right]\right\} +$$

$$+\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=n-p+1}^{n}\left[(a+1)+\frac{k}{n}\right]^2\right\} = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n}\left[(a+1)+\frac{k}{n}\right]^2\right\} =$$

$$=\int_0^1\left[(a+1)+x\right]^2 dx = \left.\frac{(a+1+x)^3}{3}\right|_0^1 = \frac{(a+2)^3-(a+1)^3}{3} = a^2+3a+\frac{7}{3}$$

Exemplul 2.5. Să se găsească limita şirului cu termenul general

$$a_n = \frac{1^k+2^k+...+n^k}{n^{k+1}}$$

Soluţie: metoda I

Amintim teorema lui Cesaro-Stolz:

Fie $(a_n)_{n\in\mathbf{N}}$ şi $(b_n)_{n\in\mathbf{N}}$ două şiruri de numere. Dacă:

1^0 şirul $(b_n)_{n\in\mathbf{N}}$ este monoton şi nemărginit;

2^0 $\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=l$

atunci avem: $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=l$.

Considerăm $a_n=1^k+2^k+...+n^k$ şi $b_n=n^{k+1}$. Avem:

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{(n+1)^k}{(n+1)^{k+1}-n^{k+1}} = \frac{(n+1)^k}{[(n+1)-n]\sum\limits_{s=1}^{k}(n+1)^{k-s}n^s} =$$

$$= \frac{n^k\left(1+\frac{1}{n}\right)^k}{\sum\limits_{s=1}^{k}n^{k-s}\left(1+\frac{1}{n}\right)^{k-s}n^s} = \frac{\left(1+\frac{1}{n}\right)^k}{\sum\limits_{s=1}^{k}\left(1+\frac{1}{n}\right)^{k-s}}$$

Deci: $\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim\limits_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^k}{\sum\limits_{s=1}^{k}\left(1+\frac{1}{n}\right)^{k-s}} = \frac{1}{k+1}$.

metoda II

$$\lim_{n\to\infty}a_n = \lim_{n\to\infty}\frac{1^k+2^k+...+n^k}{n^{k+1}} = \lim_{n\to\infty}\sum_{s=1}^{n}\frac{1}{n}\left(\frac{s}{n}\right)^k = \int_0^1 x^k dx = \left.\frac{x^{k+1}}{k+1}\right|_0^1 = \frac{1}{k+1}.$$

Digitally signed by Virgil-Mihail Zaharia