Integrala definită

Definiţie. Fie f: [a, b]$\rightarrow$**R**, o funcţie care admite primitive pe [a,b] şi F o primitivă a lui f. Numim integrala definită de la a la b a lui f, expresia:

$F(b) - F(a)$ şi notăm: $\int_a^b f(x)dx = F(b) - F(a)$ (formula lui Leibniz- Newton).

Notaţie: $F(b) - F(a) = F(x)\Big|_a^b$ (citit:"F(x) luat între a şi b).

Reamintim că orice funcţie continuă admite primitive.

Proprietăţi ale integralei definite.

1) Dacă f,g:[a,b]$\rightarrow$**R** sunt funcţii continue şi $\lambda \in$ **R**, atunci:

a) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$;

b) $\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$.

2) Dacă f: [a, b]$\rightarrow$**R**, este o funcţie pozitivă şi continuă$\Rightarrow \int_a^b f(x)dx \geq 0$.

3) Dacă f,g:[a,b]$\rightarrow$**R** sunt funcţii continue, cu proprietatea: $f(x) \leq g(x), \forall x \in [a,b]$, atunci:

$\int_a^b f(x)dx \leq \int_a^b g(x)dx$;

4) Fie f: [a, b]$\rightarrow$**R** şi $c \in (a,b)$; dacă restricţiile lui f, sunt continue pe [a,c] şi [c,b], atunci:

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Definiţie. Dacă $a \leq b$ şi f:[a,b] $\rightarrow$**R** este o funcţie continuă, atunci punem prin definiţie:

a) $\int_a^b 0dx = 0$

b) $\int_a^a f(x)dx = 0$;

c) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

Aplicaţii: Calculaţi integralele:

1) $\int_{-2}^{1} (x^3 - 3x + 4)dx$; 2) $\int_2^4 \left(x + \frac{3}{x^2}\right)dx$;

3) $\int_{-2}^{2} (|x-1| + |x+1|)dx$; 4) $\int_{-3}^{-1} (2e^x - x)dx$.

Rezolvare

$$\int_{-2}^{1}\left(x^3-3x+4\right)dx=\left(\frac{x^4}{4}-3\frac{x^2}{2}+4x\right)\Bigg|_{-2}^{1}=\frac{1^4}{4}-3\cdot\frac{1^2}{2}+4\cdot 1-\left(\frac{(-2)^4}{4}-3\cdot\left(\frac{(-2)^2}{2}\right)+4\cdot(-2)\right)=$$

1) $$=\frac{1}{4}-\frac{{}^{2)}3}{2}+^{4)}4-\left(\frac{16}{4}-\frac{12}{2}-8\right)=\frac{1-6+16}{4}-(4-6-8)=\frac{11}{4}+^{4)}10=\frac{51}{4}$$;

Calcul

$$F(x)=\int\left(x^3-3x+4\right)dx=\int x^3dx-3\int xdx+4\int dx=\frac{x^4}{4}-3\frac{x^2}{2}+4x+C$$

2) $$\int_{2}^{4}\left(x+\frac{3}{x^2}\right)dx=\int_{2}^{3}xdx+3\int_{2}^{4}\frac{1}{x^2}dx=\frac{x^2}{2}\Bigg|_{2}^{4}-3\cdot\frac{1}{x}\Bigg|_{2}^{4}=\frac{4^2}{2}-\frac{2^2}{2}-3\left(\frac{1}{4}-\frac{1}{2}\right)=$$;

$$=8-2-3\cdot\left(-\frac{1}{4}\right)=^{4)}6+\frac{3}{4}=\frac{27}{4}$$

$$\int_{-2}^{2}\left(|x-1|+|x+1|\right)dx\overset{4)}{=}\int_{-2}^{-1}(-2x)\,dx-\int_{-1}^{1}2dx+\int_{1}^{2}2xdx=-\cancel{2}\cdot\frac{x^2}{\cancel{2}}\Bigg|_{-2}^{-1}+2x\Big|_{-1}^{1}+\cancel{2}\cdot\frac{x^2}{\cancel{2}}\Bigg|_{1}^{2}=$$

$$=-(-1)^2+(-2)^2+2\cdot 1-2\cdot(-1)+2^2-1^2=\cancel{-1}+4+2\cancel{+2}+4\cancel{-1}=10$$

3) Explicit ;

$$|x-1|=\begin{cases}x-1, & x\geq 1\\ -x+1, & x<1\end{cases},\ |x+1|=\begin{cases}x+1, & x\geq -1\\ -x-1, & x<-1\end{cases}$$

$$|x-1|+|x+1|=\begin{cases}-x+1-x-1=-2x, & x<-1\\ -x+1+x+1=2, & x\in[-1,1)\\ x-1+x+1=2x, & x\geq 1\end{cases}$$

4) $$\int_{-3}^{-1}\left(2e^x-x\right)dx=2\int_{-3}^{-1}e^xdx-\int_{-3}^{-1}xdx=2e^x\Big|_{-3}^{-1}-\frac{x^2}{2}\Bigg|_{-3}^{-1}=2e^{-1}-2e^{-3}-\left(\frac{(-1)^2}{2}-\frac{(-3)^2}{2}\right)=$$.

$$=\frac{2}{e}-\frac{2}{e^3}-\left(\frac{1}{2}-\frac{9}{2}\right)=\frac{2}{e}-\frac{2}{e^3}-(-4)=\frac{2}{e}-\frac{2}{e^3}+4$$

5) $$\int_0^1\left(2^x+2^{-x}\right)dx=\int_0^1 2^x\,dx+\int_0^1\left(2^{-1}\right)^x dx=\frac{2^x}{\ln 2}\Bigg|_0^1+\frac{\left(2^{-1}\right)^x}{\ln 2^{-1}}\Bigg|_0^1=\frac{2^1-2^0}{\ln 2}-\frac{\left(2^{-1}\right)^1-\left(2^{-1}\right)^0}{\ln 2}=$$

$$=\frac{1}{\ln 2}\left(2\cancel{-1}-\frac{1}{2}\cancel{+1}\right)=\frac{3}{2}\cdot\frac{1}{\ln 2}$$

Folosind formula lui **Leibniz –Newton** calculaţi următoarele integrale:

1) $\int_0^{\sqrt{2}} x^5 dx = \frac{4}{3}$;

2) $\int_{\sqrt{2}}^{\sqrt{3}} x^5 dx = \frac{19}{6}$;

3) $\int_{-3}^{3} x^7 dx = 0$;

4) $\int_{-\sqrt{5}}^{-1} x^7 dx = -78$;

5) $\int_0^4 \sqrt{x\sqrt{x}}\, dx = \frac{4 \cdot 2^{\frac{7}{2}}}{7}$;

6) $\int_{-4}^{-1} \frac{1}{\sqrt[3]{x}} dx = \frac{3-3(-4)^{\frac{2}{3}}}{2}$;

7) $\int_{-8}^{-1} \frac{1}{\sqrt[3]{x}} dx = -\frac{9}{2}$;

8) $\int_{-2}^{-1} \frac{x^2+1}{x} dx = -\frac{3}{2} - \ln 2$;

9) $\int_{-1}^{1} e^x dx = e - \frac{1}{e}$;

10) $\int_1^2 10^x dx = \frac{90}{\ln 10}$;

11) $\int_0^{\sqrt{2}} \left(\sqrt{2}\right)^x dx = \frac{2}{\ln 2}\left(2^{\frac{1}{\sqrt{2}}} - 1\right)$

12) $\int_{-4}^{4} \frac{1}{x^2-25} dx = -\frac{2\ln 3}{5}$;

13) $\int_{-1}^{3} \frac{1}{x^2+3} dx = \frac{\pi\sqrt{3}}{6}$

14) $\int_0^{\pi} \sin x dx = 2$;

15) $\int_0^{\frac{\pi}{2}} \cos x dx = 1$;

16) $\int_{\frac{3\pi}{4}}^{\pi} \frac{1}{\cos^2 x} dx = 1$;

17) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x} dx = \sqrt{3}$;

18) $\int_0^{2\sqrt{2}} \frac{1}{\sqrt{x^2+1}} dx = \ln\left(2\sqrt{2}+3\right)$;

19) $\int_{-3}^{-2} \frac{1}{\sqrt{x^2-3}} dx = -\ln\left(3-\sqrt{6}\right)$

20) $\int_{-\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} dx = \frac{7\boldsymbol{\pi}}{12}$;

21) $\int_0^{\frac{3}{4}} \frac{1}{\sqrt{9-4x^2}} dx = \frac{\boldsymbol{\pi}}{12}$;

22) $\int_0^1 \left(3\sqrt{x} + 4\sqrt[3]{x}\right) dx = 5$;

23) $\int_1^e \left(x^5 - 4x^3 + \frac{3}{x} + \frac{2}{x^2}\right) dx = \frac{\boldsymbol{e}^6}{6} - \frac{2}{\boldsymbol{e}} - \boldsymbol{e}^4 + \frac{35}{6}$

24) $\int_4^5 \left(\frac{6}{x^2-9} - \frac{3}{\sqrt{x^2-9}}\right) dx = 3\ln\left(\sqrt{7}+4\right) + \ln(7) - 2\ln(2) - 6\ln(3)$

Integrarea prin părţi

Teoremă. Dacă $f,g;[a,b]\to\mathbf{R}$, sunt două funcţii derivabile cu derivate continue,

atunci are loc relaţia: $\int_a^b f'(x)g(x)dx = f(x).g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx$.(1)

(*formula de integrare prin părţi*)

Aplicaţii: Calculaţi, folosind formula de integrare prin părţi:

1) $\int_0^2 (x+1)\cdot e^x dx = \int_0^2 (e^x)'(x+1)dx = e^x(x+1)\Big|_0^2 - \int_0^2 e^x(x+1)'dx = e^2\cdot 3 - 1 - \int_0^2 e^x dx =$

$= 3e^2 - 1 - e^x\Big|_0^2 = 3e^2 - 1 - (e^2 - e^0) = 3e^2 - 1 - e^2 + 1 = 2e^2$.

2) $\int_1^e \ln x dx = \int_1^e (x)'\ln x dx = x\cdot\ln x\Big|_1^e - \int_1^e x\cdot\frac{1}{x}dx = e - x\Big|_1^e = e-(e-1) = e-e+1 = 1.$

3)

$I = \int_0^\pi e^{2x}\sin 3x dx = \underbrace{\frac{e^{2x}}{2}\cdot\sin 3x\Big|_0^\pi}_{=0} - \int_0^\pi \frac{e^{2x}}{2}\cdot 3\cos 3x dx = -\frac{3}{2}\int_0^\pi e^{2x}\cdot\cos 3x dx =$

$= -\frac{3}{2}\left(\frac{e^{2x}}{2}\cos 3x\Big|_0^\pi - \int_0^\pi \frac{e^{2x}}{2}\cdot(-3\sin 3x)dx\right) = -\frac{3}{2}\cdot\frac{e^{2\pi}}{2} + \frac{3}{4} - \frac{9}{4}\int_0^\pi e^{2x}\cdot\sin 3x dx \Rightarrow$

$I = \frac{e^{2\pi}}{4} + \frac{3}{4} - \frac{9}{4}I \Rightarrow \frac{13}{4}I = \frac{e^{2\pi}+3}{4} \Rightarrow I = \frac{e^{2\pi}+3}{13}$.

4)

$I = \int_4^5 \sqrt{x^2+4}dx = \int_4^5 \frac{x^2+4}{\sqrt{x^2+4}}dx = \int_4^5 x\cdot\frac{x}{\sqrt{x^2+4}}dx + 4\int_4^5 \frac{1}{\sqrt{x^2+4}}dx =$

$= \int_4^5 x\cdot\left(\sqrt{x^2+4}\right)' dx + 4\ln\left(x+\sqrt{x^2+4}\right)\Big|_4^5 = x\cdot\sqrt{x^2+4}\Big|_4^5 - \int_4^5 \sqrt{x^2+4}dx + 4\ln\frac{5+\sqrt{29}}{4+2\sqrt{5}}$

$2I = 5\sqrt{29} - 8\sqrt{5} + 4\ln\frac{5+\sqrt{29}}{4+2\sqrt{5}} \Rightarrow I = 2\ln\frac{\sqrt{29}+5}{2\sqrt{5}+4} + \frac{5\sqrt{29}}{2} - 4\sqrt{5}$.

5) $\int_0^1 x^2e^x dx \overset{(1)}{=} \int_0^1 x^2(e^x)'dx = x^2e^x\Big|_0^1 - \int_0^1 2xe^x dx = e - 2\int_0^1 x(e^x)'dx = e - 2\left(xe^x\Big|_0^1 - \int_0^1 e^x dx\right) =$

$= e - 2(e-e+1) = e-2$

6) $\int_1^2 x\ln x dx \overset{(1)}{=} \int_1^2\left(\frac{x^2}{2}\right)'\ln x dx = \frac{x^2}{2}\ln x\Big|_1^2 - \frac{1}{2}\int_1^2 x^{\not 2}\frac{1}{\not x}dx = 2\ln 2 - \frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^2 = 2\ln 2 - \frac{3}{4}$.

7) $\int_0^\pi x\sin x dx \overset{(1)}{=} \int_0^\pi x(-\cos)' x dx = -x\cos x\Big|_0^\pi + \int_0^\pi \cos x dx = \pi$.

8) $\int_1^e \frac{\ln x}{x^2}dx \overset{(1)}{=} \int_1^e \ln x\left(\frac{-1}{x}\right)' dx = -\frac{1}{x}\ln x\Big|_1^e + \int_1^e \frac{1}{x^2}dx = -\frac{1}{e}\ln e + \ln 1 - \frac{1}{x}\Big|_1^e = -\frac{1}{e} - \frac{1}{e} + 1 = \frac{e-2}{e}$.

Calculaţi:

1. $\int_0^1 x^2 e^x dx = e^x(2-2x+x^2)\Big|_0^1 = \boldsymbol{e}-2$

2. $\int_1^2 x\ln x dx = \left(-\frac{x^2}{4}+\frac{1}{2}x^2\ln x\right)\Bigg|_1^2 = 2\ln 2-\frac{3}{4}$

3. $\int_1^2 x^2\ln x dx = \left(-\frac{x^3}{9}+\frac{1}{3}x^3\ln x\right)\Bigg|_1^2 = \frac{8\ln 2}{3}-\frac{7}{9}$

4. $\int_0^\pi x\sin x dx = (-x\cos x+\sin x)\Big|_0^\pi = \pi$

5. $\int_1^e \frac{\ln x}{x^2}dx = \left(-\frac{1}{x}-\frac{\ln x}{x}\right)\Bigg|_1^e = 2$

6. $\int_0^\pi x^2\cos x dx = \left(2x\cos x+(-2+x^2)\sin x\right)\Big|_0^\pi = -2\pi$

7. $\int_4^5 \sqrt{x^2-9}dx = \left[\frac{1}{2}x\sqrt{x^2-9}-\frac{9}{2}\ln\left(x+\sqrt{x^2-9}\right)\right]\Bigg|_4^5 = \frac{9\ln\left(\sqrt{7}+4\right)}{2}+10-2\sqrt{7}-9\ln 3$

8. $\int_0^{\frac{\pi}{4}} xtg^2 xdx = \left(-\frac{x^2}{2}+xtgx+\ln(\cos x)\right)\Bigg|_0^{\frac{\pi}{4}}$

9. $\int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x}dx = [xtgx+\ln(\cos x)]\Big|_0^{\frac{\pi}{4}} = \ln\left(\cos\frac{\pi}{4}\right)+\frac{1}{4}\pi tg\frac{\pi}{4} = \frac{\pi}{4}-\frac{\ln 2}{2}$

10. $\int_0^\pi x^2\sin^2 xdx = \frac{1}{24}\left(4x^3+\left(3-6x^2\right)\sin 2x-6x\cos 2x\right)\Big|_0^\pi = \frac{1}{24}(4\pi^3-6\pi\cos 2\pi+(3-6\pi^2)\sin 2\pi) = \frac{1}{24}\left(-6\pi+4\pi^3\right)$

11. $\int_0^{\frac{\sqrt{3}}{2}} \arcsin xdx = \left(\sqrt{1-x^2}+x\arcsin x\right)\Big|_0^{\frac{\sqrt{3}}{2}} = \frac{1}{6}(-3+\sqrt{3}\pi)$

12. $\int_0^1 arctg\sqrt{2x+1}dx = \left(-\frac{1}{2}\sqrt{1+2x}+(1+x)\text{arctg}\sqrt{1+2x}\right)\Bigg|_0^1 = \frac{1}{12}(6-6\sqrt{3}+5\pi)$

13. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} xctg^2 xdx = \left(-\frac{x^2}{2}-x\,\text{ctg}\,x+\ln(\sin x)\right)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{288}\left[\pi\left(72-32\sqrt{3}-7\pi\right)+144\ln\frac{3}{2}\right]$

14. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{x\cos x}{\sin^2 x}dx = \frac{1}{18}\left[\left(-9+4\sqrt{3}\right)\pi+9\ln[3]\right]$

15. $\int_0^{\pi} e^x \cos^3 x dx = \frac{1}{40} e^{\pi}\left[3\left(5\sin\pi + \sin 3\pi\right) + 15\cos\pi + \cos 3\pi\right] - \frac{2}{5} =$
$= \frac{1}{40} e^{\pi}\left(-15-1\right) - \frac{2}{5} = -\frac{2}{5}\left(e^{\pi}+1\right)$

16. $\int_1^e \frac{x\ln x}{(x^2+1)^2} dx$

17. $\int_0^1 \frac{x^2}{(x+2)^2} e^x dx$

18. $\int_0^1 (x^2-2x+3)e^x dx$

19. $\int_0^{\pi} 2x^2 \sin^2 \frac{x}{2} dx$

20. $\int_0^e 2x\ln(x^2+1)dx$

21. $\int_1^e \frac{x\ln x}{(x^2+1)^2} dx$

22. $\int_0^1 \frac{1-x}{1+x} \ln(1+x)dx$

23. $\int_0^{\frac{1}{2}} x\ln\frac{1+x}{1-x} dx$

24. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\text{xcosx}}{\sin^2 x} dx$

25. $\int_0^{\frac{1}{2}} x\arcsin x dx$

26. $\int_1^3 arctg\sqrt{x} dx$

27. $\int_0^1 \sqrt{x^2+1} dx$

28. $\int_{\frac{1}{e}}^1 \sin(\ln x)dx$

29. $\int_{\frac{1}{e}}^1 \cos(\ln x)dx$

30. $\int_{e}^{e^2} \frac{\ln x}{x} dx$